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Free vibration of layered truncated conical shell frusta of differently varying thickness by the method of collocation with cubic and quintic splines

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Abstract

Free vibrations of layered conical shell frusta of differently varying thickness are studied using the spline function approximation technique. The equations of motion for layered conical shells, in the longitudinal, circumferential and transverse displacement components, are derived using extension of Love's first approximation theory. Assuming the displacement components in a separable form, a system of coupled equations on three displacement functions are obtained. Since no closed form solutions are generally possible, a numerical solution procedure is adopted in which the displacement functions are approximated by cubic and quintic splines. A generalized eigenvalue problem is obtained which is solved numerically for an eigenfrequency parameter and an associated eigenvector of spline coefficients. The vibrations of two-layered conical shells, made up of several types of layer materials and supported differently at the ends are considered. Linear, sinusoidal and exponential variations in thickness of layers are assumed. Parametric studies are made on the variation of frequency parameter with respect to the relative layer thickness, cone angle, length ratio, type of thickness variation and thickness variation parameter. The effect of neglecting the coupling between bending and stretching is also analysed.

Keywords: Free vibration; Conical shell; Variable thickness; Collocation; Cubic and quintic splines

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1. Introduction

Conical shell structures find wide ranging applications in areas including aerospace industry, ship construction and chemical industry. Desirable improvement in damping and shock absorbing characteristics is possible with layering of the shell walls. A concise review of material available on mechanics of composite structures is provided by Bert and Francis (1974) while the survey of Chang (1981) is exclusively on vibration of conical shells. Irie et al. (1982, 1984) studied free vibration of conical shells with constant and variable thickness. Sankaranarayanan et al. (1988) considered layered conical shells of linearly varying thickness, using Rayleigh–Ritz method of solution. The versatile numerical method FEM was used by Sivadas and Ganesan (1991) to study the vibration of laminated conical shells with variable thickness. Shu (1996) presented a generalized differential quadrature method for the vibration analysis of laminated conical shells. Wu and Wu (2000) provided 3-D elasticity solutions for the free vibration analysis of laminated conical shells by an asymptotic approach. Viswanathan and Navaneethakrishnan (2003) recently made a spline fucntion study of free vibration of layered cylindrical shells. Any variation in thickness in all the above was only linear.

In the current work the free vibration of conical shell frusta made up of layers of varying thickness is analyzed using a spline function collocation technique. The thickness variation is not restricted to be linear.

The problem is formulated by extending Love's first approximation theory on homogeneous shells. The layers are considered to be thin, elastic and specially orthotropic or isotropic and assumed to be bonded perfectly together and to move without interface slip. They are assumed to be of variable thickness, the variation being linear, exponential and sinusoidal. The governing differential equations of motions are obtained in terms of the reference surface displacement. Different types of layer material are considered and three sets of boundary conditions are imposed.

The differential equations of motion obtained are coupled in the longitudinal, circumferential and transverse displacement components. They reduce to a system of ordinary differential equations on a set of assumed displacement functions which are functions of the meridional co-ordinate only. The equations have no closed form solution in general. Numerical solution techniques have to be resorted to. A spline function approximation technique is used here in preference to several other methods, since in this a chain of lower order approximations is used which can yield greater accuracy than a global higher order approximation. This conjuncture was made and tested by Bickley (1968) over a two point boundary value problem with a cubic spline. Subsequently, Soni and Sankara Rao (1974), Irie et al. (1979), Irie and Yamada (1980), Navaneethakrishnan (1988, 1993), Navaneethakrishnan et al. (1992) and Viswanathan and Navaneethakrishnan (2002, 2003) have also demonstrated this, but most of them used only a single spline function in a problem.

In this study displacement functions are approximated by splines which are cubic or quintic, in a system of coupled equations. These splines are simple and clear for analytical process and therefore have significant computational advantage. The differential equations are modified and collocation procedure is applied to obtain a set of field equations, which along with the equations of boundary conditions yield a system of homogeneous simultaneous algebraic equations on the assumed spline co-efficients. The resulting generalized eigenvalue problem is solved for a frequency parameter using eigensolution technique to get as many eigenfrequencies as required, starting from the least. From the eigenvectors the spline co-efficients are computed from which the mode shapes are constructed. Extensive parametric studies are made for two layered shells. The effect of the length ratio, the relative thickness of the layers, the overall thickness-to-radius ratio, the taper ratio, the thickness variation parameters and the vertical angle of the conical shell on the frequency parameter are analyzed. Significant mode shapes are presented. The results are presented in terms of graphs and discussed.

2. Formulation

The system of differential equations characterizing the vibration of a thin shell of revolution, comprising of isotropic and specially orthotropic layers, is derived. The general line of procedure of Ambartsumyan (1964) for the classical theory of thin shell is adopted. The development is based on the Love's first approximation theory in which the rotatory inertia and transverse shear deformation are neglected. Such an approach results in analytically simpler procedure, by way of less number of equations of motion and avoidance of nonlinear terms in them, conveniencing the application of spline function method. Considering these gains, the little loss in accuracy of results may be well tolerated. The coordinate system and the geometric parameters for the truncated conical shell for linearly varying thickness are shown in Fig. 1. The thickness of the kth layer is taken in the form

$$h_k(x) = h_{ok}g(x) \tag{1}$$

where h_{ok} is a constant. When $g(x) = 1$ the thickness becomes uniform. For variable thickness $g(x)$ is a suitable function of x.

If z_k is the distance of the lower edge of the kth layer from the reference surface and ρ_k is the mass density of the material of the kth layer, then

$$\sum_k (z_k^2 - z_{k-1}^2)\rho_k = 0 \tag{2}$$

where ρ_k is the density of the kth layer and z_k is the distance of the outer boundary of the kth layer from the reference surface.This may be interpreted as determining one of the z_k in terms of the rest of the z_k.

If the shell wall has only two layers, in particular, one can obtain

$$\begin{aligned} z_o(x) &= z_{oo}g(x) \\ z_1(x) &= z_o(x) + h_1(x) = z_{o1}g(x) \\ z_2(x) &= z_1(x) + h_2(x) = z_{o2}g(x) \end{aligned} \tag{3}$$

Here $z_{ok} = z_{ok-1} + h_{ok-1}$. It may be noted that for linear and sinusoidal variation of this thickness $z_{ok} = z_k(a)$. Denoting the elastic coefficients corresponding to layers of uniform thickness with superscript 'c', one easily finds

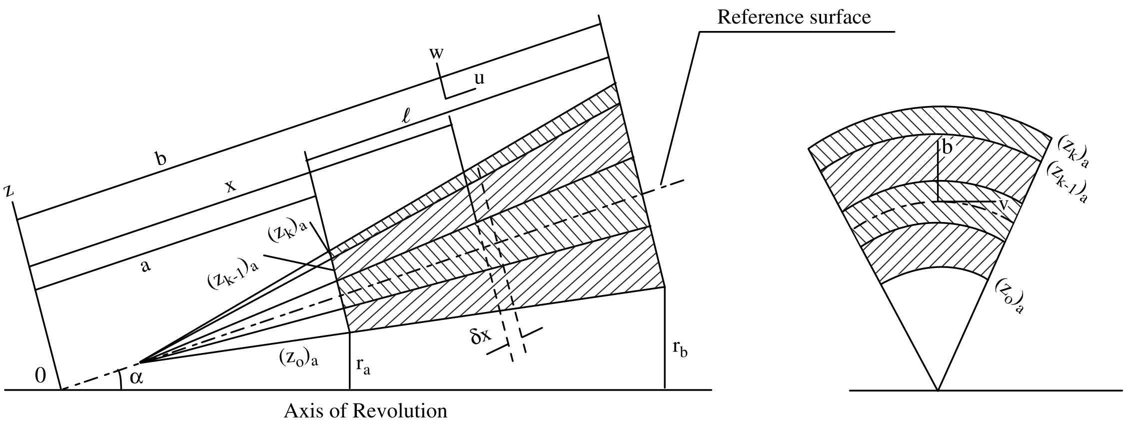

Fig. 1. General arrangement of layers of linearly variable thickness.

$$A_{ij} = A^c_{ij}g(x), \quad B_{ij} = B^c_{ij}g(x), \quad D_{ij} = D^c_{ij}g(x), \tag{4}$$

For our detailed study, the thickness variation of each layer is assumed in the form

$$h(x) = h_o g(x) \tag{5}$$

where

$$g(x) = 1 + C_\ell\left(\frac{x - x_a}{\ell}\right) + C_e \exp\left(\frac{x - x_a}{\ell}\right) + C_s \sin\left(\frac{\pi(x - x_a)}{\ell}\right) \tag{6}$$

Here ℓ is the length of the cone and x_a is the distance from the origin to $x = a$.

The stress resultants and moment resultants are expressed in terms of the longitudinal, circumferential and transverse displacements u, v and w of the reference surface. The displacements are assumed in the separable form given by

$$\begin{aligned} u(x,\theta,t) &= U(x)\cos n\theta \mathrm{e}^{\mathrm{i}\omega t} \\ v(x,\theta,t) &= V(x)\sin n\theta \mathrm{e}^{\mathrm{i}\omega t} \\ w(x,\theta,t) &= W(x)\cos n\theta \mathrm{e}^{\mathrm{i}\omega t} \end{aligned} \tag{7}$$

where x and θ are the longitudinal and rotational co-ordinates, t is the time, ω is the angular frequency of vibration and n is the circumferential mode number.When $n = 0$, the vibration becomes axisymmetric. Using Eq. (7) in the constitutive equations and the resulting expressions for the stress resultants and the moment resultants in the equilibrium equations, (given in Appendix A) the governing differential equations of motion are obtained in the form

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \{0\} \tag{8}$$

The operators $L_{ij}(i,j = 1,2,3)$ are defined in Appendix B.

3. Method of solution

3.1. Modification of displacement equations

The differential equations on the displacement functions derived in the last section contain derivatives of third order in U, second-order in V and fourth order in W. As such they are not amenable to the solution procedure we propose to adopt. Hence the equations are combined within themselves and a modified set of equations are derived.

The procedure adopted to this end is to differentiate the first of Eq. (8) with respect to x once and to use it to eliminate $U'''(x)$ in the third equation. The modified set of equations are of order 2 in U, order 2 in V and order 4 in W, given by

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L^*_{31} & L^*_{32} & L^*_{33} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = \{0\} \tag{9}$$

in which the new operators are L^*_{31}, L^*_{32} and L^*_{33} whose complete forms are not given here for want of space.

3.2. Transformation

The parameters are non dimensionalized by writing

$$\begin{aligned} &\lambda = \ell\lambda', \quad \text{a frequency parameter} \\ &\beta = a/b, \ \text{a length ratio} \\ &\gamma = h_o/r_a, \gamma' = h_o/a, \quad \text{ratios of thickness to radius and to a length and} \\ &\delta_k = h_k/h, \quad \text{a relative thickness ratio} \end{aligned} \tag{10}$$

Here h_k is the thickness of the kth layer, h is the total thickness of the shell, h_o is the constant thickness and r_a is the radius of small end of the cone. When there are only two layers, we define $\delta = \delta_1$, and hence $\delta_2 = 1 - \delta$.

For a two-layered shell, the independent geometric material parameters are α, β, γ (or γ'), δ, C_ℓ, C_e and C_s.

The meridional coordinate of reference is non-dimensionalized with the transformation

$$X = \frac{x-a}{\ell}, \quad a \leqslant x \leqslant b \tag{11}$$

Clearly $X \in [0,1]$, which makes the form of splines assumed still more elegant.

3.3. Thickness variation

The thickness h_k (X) of the kth layer at the point distant X from the smaller end of the cone, already explained, can be expressed as

$$h_k(X) = h_{ok}g(X) \tag{12}$$

where

$$g(X) = 1 + C_\ell X + C_e \exp(X) + C_s \sin \pi X \tag{13}$$

Case (i)
If $C_e = C_s = 0$, the thickness variation becomes linear. In this case it can be easily shown that

$$C_\ell = \frac{1}{\eta} - 1 \tag{14}$$

where η is the taper ratio $h_k(0)/h_k(1)$.

The case $\eta = \beta$ corresponds to the special case in which the thickness at any point is proportional to the distance of the point from the vertex of the cone.

If $\eta = 1$, then $C_\ell = 0$ and the thickness becomes constant.

Case (ii)
If $C_\ell = C_s = 0$, the excess thickness over uniform thickness varies exponentially.

Case (iii)
If $C_\ell = C_e = 0$, the excess thickness varies sinusoidally.

It may be noted that the thickness of any layer at the end $X = 0$ is h_{ok} for the cases (i) and (iii), but is h_{ok} $(1 + C_e)$ for the case (ii).

The following range of values of the thickness coefficients are considered:

$$0.5 \leqslant \eta \leqslant 2.1, \quad -0.2 \leqslant C_e \leqslant 0.2, \quad -0.5 \leqslant C_s \leqslant 0.5 \tag{15}$$

Though the thickness variation can be more general, the types of variations considered here are typical and realizable. Since several functions can be reasonably approximated by a few terms of their Fourier series, the sinusoidal variation is typical.

3.4. Spline collocation procedure

The range of X, namely $[0,1]$ is divided into N subintervals, at the points $X = X_s$, $s = 1,2,\ldots,N-1$. Having these points along with end points of the subintervals for the knots, we make the following spline function approximations to the displacement functions $U(X)$, $V(X)$ and $W(X)$, respectively.

$$\begin{aligned}
U^*(X) &= \sum_{i=0}^{2} a_i X^i + \sum_{j=0}^{N-1} b_j (X - X_j)^3 H(X - X_j) \\
V^*(X) &= \sum_{i=0}^{2} c_i X^i + \sum_{j=0}^{N-1} d_j (X - X_j)^3 H(X - X_j) \\
W^*(X) &= \sum_{i=0}^{4} e_i X^i + \sum_{j=0}^{N-1} f_j (X - X_j)^5 H(X - X_j)
\end{aligned} \tag{16}$$

Here, $H(X - X_j)$ is the Heaviside step function. In these U^*, V^* are cubic spline and W^* is quintic. They are in the minimum degree required, since the system of differential Eq. (9) are of second degree in U and V and of fourth degree in W.

The width of each subinterval is $1/N$ and $X_s = s/N$, $s = 0,1,\ldots,N$, since the knots X_s are chosen equally spaced.

Since the splines assumed must agree with functions they are approximating at the nodes (which coincide with the knots here), we require that these splines satisfy the differential equations given by Eq. (9), at all X_s. This results in the homogeneous system of $(3N+3)$ equations in the $(3N+11)$ spline coefficients, a_i, c_i, b_j, d_j, e_k, f_j ($i = 0,1,2$; $k = 0,1,2,3,4$; $j = 0,1,2,\ldots,N-1$).

The following boundary conditions are used: (i) both the edges clamped (C–C), (ii) both the edges hinged (H–H) and (iii) smaller edge clamped and the other edge free (C–F). Each of these cases furnishes eight more equations thus giving in each case, a total of $(3N+11)$ equations, in the same number of unknowns. The resulting field and the boundary condition equations can be represented as

$$[M]\{q\} = \lambda^2 [P]\{q\} \tag{17}$$

where $[M]$ and $[P]$ are matrices of order $(3N+7)\times(3N+7)$ and $\{q\}$ is the matrix of order $(3N+7)\times 1$. This is treated as a generalized eigenvalue problem in the eigen parameter λ and the eigenvector whose elements are the spline coefficients.

4. Convergence and comparative studies

Since the matrices of large orders were to be handled, double precision arithmetic was used throughout for numerical computations. The material properties are taken from Elishakoff and Stavsky (1976).

Extensive trial runs of the computer program developed were carried out analyzing the convergence of the frequency parameter values with the number of subintervals N of the range of X. The program was run for several cases of parametric values, material combinations, thickness variations and reduced cases, for values of $N = 4$ onwards. The computed values of λ improved with increase of N, but the improvement came down steadily. It can be seen that the choice of $N = 14$ is adequate since for the next value of N the percent changes in values of λ_1 are very low, the maximum being 0.35%.

Table 1
Comparative study of vibration of homogeneous conical shell of linear variation in thickness under the C–F boundary conditions with Irie et al. (1982) and Sankaranarayanan et al. (1987) $\nu = 0.3$, $\gamma = 0.01$, $\alpha = 30°$, $\beta = 0.5$

m	λ_m		
	Present	Irie et al.	Sankaranarayanan et al.
1	12.9832	12.99	13.00
2	15.871	15.88	15.89
3	17.5673	17.57	17.57

Table 2
Comparative study of vibration of homogeneous conical shell of constant thickness under the C–C boundary conditions with Irie et al. (1984) $\nu = 0.3$, $\gamma = 0.01/\beta$, $\alpha = 60°$ axisymmetric ($n = 0$) and asymmetric ($n = 4$) cases

Fundamental frequency parameter	$\beta = 0.2$		$\beta = 0.5$		$\beta = 0.75$	
	$n = 0$	$n = 4$	$n = 0$	$n = 4$	$n = 0$	$n = 4$
λ	0.5800	0.2741	0.3864	0.2545	0.2769	0.2707
λ'	0.6697	0.3165	0.6693	0.4408	0.9593	0.9376
λ''	0.6680	0.3155	0.6685	0.4298	0.9576	0.9336

λ: present value; λ': present value converted to Irie's parameter; λ'': Irie's parameter; $\lambda' = \lambda(\sin\alpha)/(1 - \beta)$.

In Table 1 the first three frequency parameter values, obtained in the work for the reduced cases of homogeneous conical shell of linear variation in thickness, with C–F boundary condition of axisymmetric case, are compared with the results of Irie et al. (1982), and Sankaranarayanan et al. (1987). These results agree very well.

In Table 2 comparison of fundamental frequency parameter values obtained presently and obtained by Irie et al. (1984) for a conical homogeneous shell of constant thickness with C–C boundary condition of axisymmetric case for three cases of length ratios are made. The agreement is quite good.

5. Results and discussion

5.1. Axisymmetric vibrations

Axisymmetric vibrations of two layered shells of different types of material combinations are considered. The variation of frequency parameter values with respect to relative layer thickness, cone angle and length ratio are investigated. The linear, exponential and sinusoidal variations in thickness of layers are taken into consideration.

The effects of altering the relative layer thickness on the frequency parameter, while the other geometric parameters are held fixed, under different boundary conditions are depicted in Figs. 2–5. Fig. 2(a) corresponds to a shell whose inner and outer layers are made up of HSG and SGE materials, respectively. Thus, when $\delta = 0$, the inner layer disappears and the shell in homogeneous, made up of SGE; when $\delta = 1$, it is again homogeneous, made up of HSG. For $0 < \delta < 1$, both the layers are present. Both the ends of the cone are clamped (C–C). Thickness of the layers vary linearly ($C_\ell \neq 0$, $C_e = C_s = 0$). The taper ratio η, which is the ratio of the thickness of the shell at $x = a$ to its thickness at $x = b$, is 0.5. The semi cone angle α is 30°. The length ratio $\beta(=a/b)$ is 0.5; we call such a shell to be of medium length, compared to short shells (large β) and long shells (small β). The thickness parameter γ is set equal to 0.05. The figure contains plots of λ_m ($m = 1, 2, 3$), where m is the meridional mode number, against δ. The continuous and dashed lines

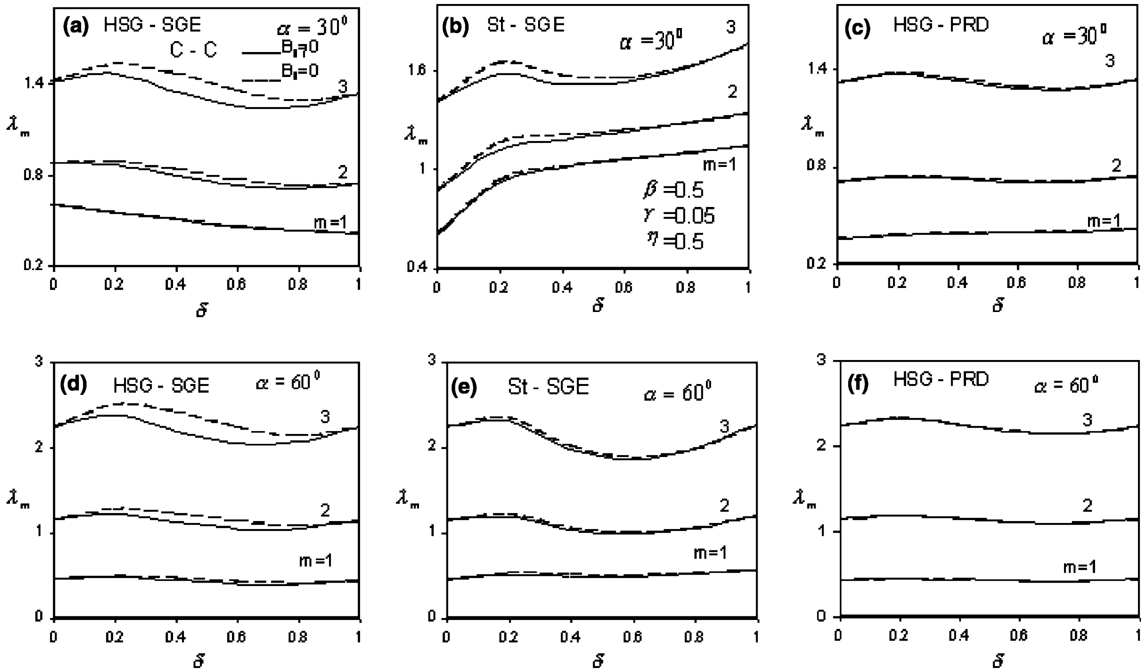

Fig. 2. Variation of frequency parameter with relative layer thickness and the effect of coupling: conical shells of linear variation in thickness of layers under clamped–clamped boundary conditions.

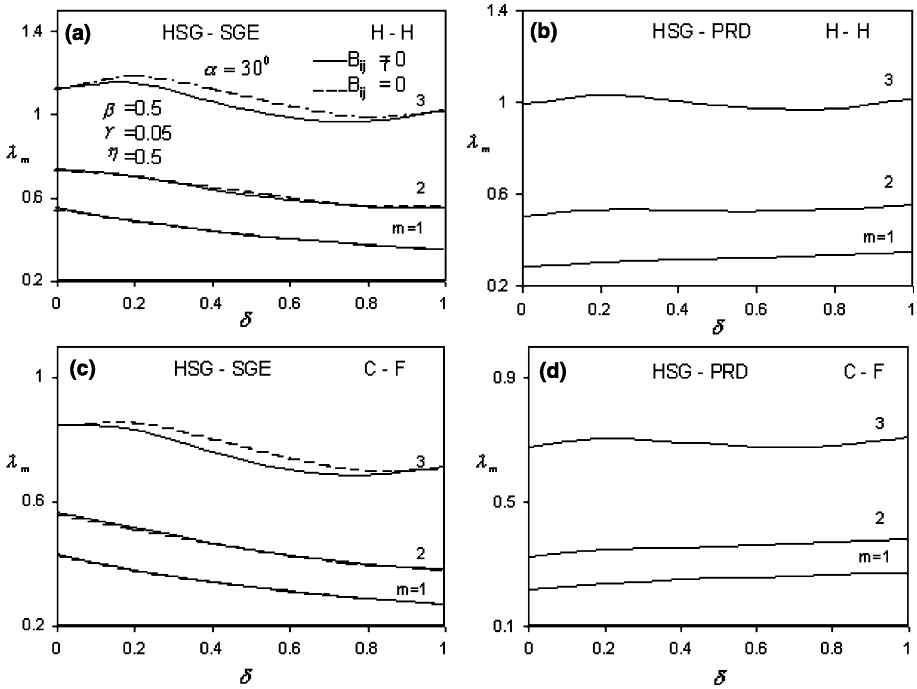

Fig. 3. Variation of frequency parameter with relative layer thickness and the effect of coupling: linear variation in thickness of layers with hinged–hinged and clamped–clamped boundary conditions.

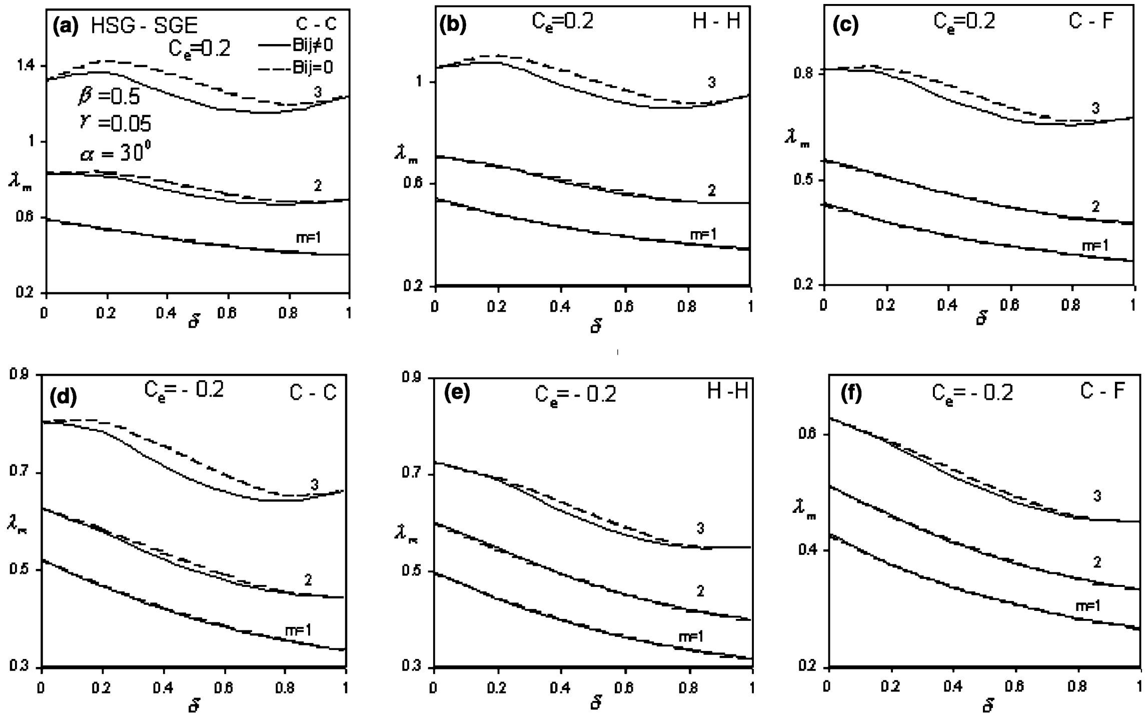

Fig. 4. Variation of frequency parameter with relative layer thickness and the effect of coupling: exponential variation in thickness of layers. Layer materials: HSG-SGE.

correspond respectively to the inclusion and neglect of the coupling effect between the longitudinal and flexural deflections, characterized by taking $B_{ij} \neq 0$ and $B_{ij} = 0$, respectively.

At the outset the variation in frequencies is seen to be very much dependent on the material of the layers, the support conditions, the nature of the thickness variation of the layers and the geometric parameters. Generally, the frequency parameter variation curves corresponding to the lowest meridional mode $m = 1$ have the least undulations. As m grows, the undulations get more and more pronounced.

It is interesting to remark that the various combinations of steel and aluminium tend to produce almost very close frequency parameter values (figures not given). The maximum percentage variations for $m = 1,2$ and 3 for St–Al shells of constant thickness were seen to be 2.00, 3.19 and 6.76, respectively. (It can be noted that $\omega \propto \lambda$ if β is constant.) The two materials are isotropic. Since they also have nearly equal E/ρ ratios, the lamination seems to affect the vibrational behaviour very little. In the case of HSG-PRD laminations, the layers are orthotropic and with the ratios E_x/ρ comparable, though not nearly same; and hence the frequencies are not much affected by relative thickness of layers. (Figs. 2(c) and (f) and 3(b) and (d)). For other material combinations (HSG-SGE, St-SGE, etc.) pronounced variations in frequencies are observed. In the case of St-SGE shells, though steel and SGE possess very close E/ρ ratios (2.696×10^7 m^{2}s^{-2} and 2.530×10^7 m^{2}s^{-2} and respectively) yet the variation in frequency is considerable, perhaps due to the effect of material orthotropy in the case of SGE.

It is significant to observe that by a proper choice of δ it is possible to achieve a desired frequency of vibration, between some bounds, which may even be higher or lower than the frequencies corresponding to shells made of either of the two materials.

In Figs. 2, 3(a) and (c), 4 and 5 are depicted the variations of λ with δ, along with the effect of including and neglecting the coupling between extensional and flexural vibrations. For St–A1 combination (not

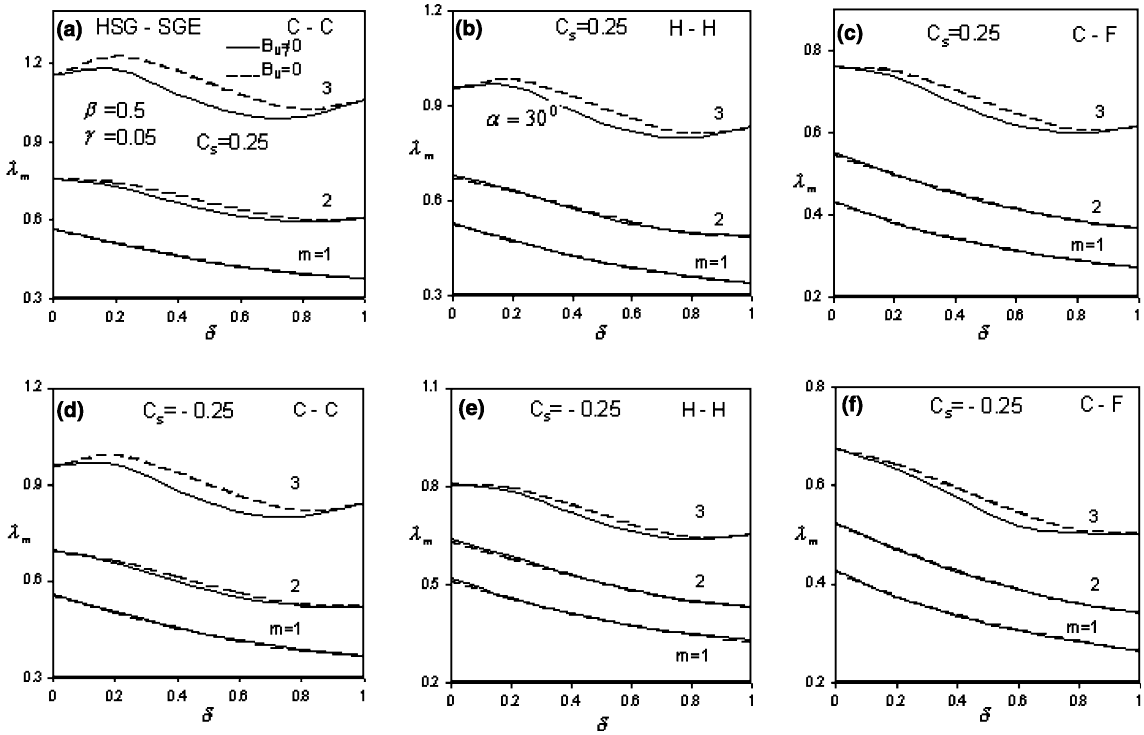

Fig. 5. Variation of frequency parameter with relative layer thickness and the effect of coupling: sinusoidal variation in thickness of layers. Layer materials: HSG-SGE.

shown) the coupling induces no change in frequencies since E/ρ ratios are nearly equal for these materials and the materials are isotropic. In the other cases it is seen as a common feature that the neglect of this coupling results in increase of the values of the frequencies. The maximum change occurs for $0.2 < \delta < 0.6$. For nearly homogeneous conditions ($\delta \leqslant 0.1$ and $\delta \geqslant 0.9$) the differences are much less, vanishing for homogeneous materials ($\delta = 0$ or 1).

Though the omission of coupling increases the frequencies in general, the maximum percent increases (e.g., 1.57%, 6.0% and 8.89% in λ_1, λ_2 and λ_3, respectively, in the case pertaining to Fig. 2(a)) is negligibly small. It shows that, with a view to reduce computational effort, without sacrificing accuracy of results significantly, one can set $B_{ij} = 0$.

Analyzing with reference to the types of boundary conditions, it is seen, as expected, that frequencies are highest for C–C conditions, lowest for C–F conditions and in-between the two for H–H conditions.

Fig. 4 correspond to exponential variations in thickness of layers ($C_\ell = C_s = 0$; $C_e \neq 0$). The cases $C_e = 0.2$ and -0.2 are studied. Accordingly the thickness of the layers at any point is higher or lower than the thickness at $x = a$. The frequencies as seen in the corresponding figures are correspondingly higher and lower. Similar remarks apply to Fig. 5, which pertain to sinusoidal variation in thickness. The meridional section of the layers are convex or concave according as $C_s = \pm 0.25$. Care has been taken to limit the range of values of these parameters so that the thickness does not vanish or become negative any where and the thin shell assumptions are valid.

In Fig. 6(a)–(c), the manner of variation of the frequency parameter with respect to the semi cone angle is studied. The layer combination considered is HSG-PRD. Fig. 6(a) and (b) pertain to shells with linearly varying thickness with the boundary conditions C–C and H–H respectively. The taper ratio $\eta = 0.7$ is

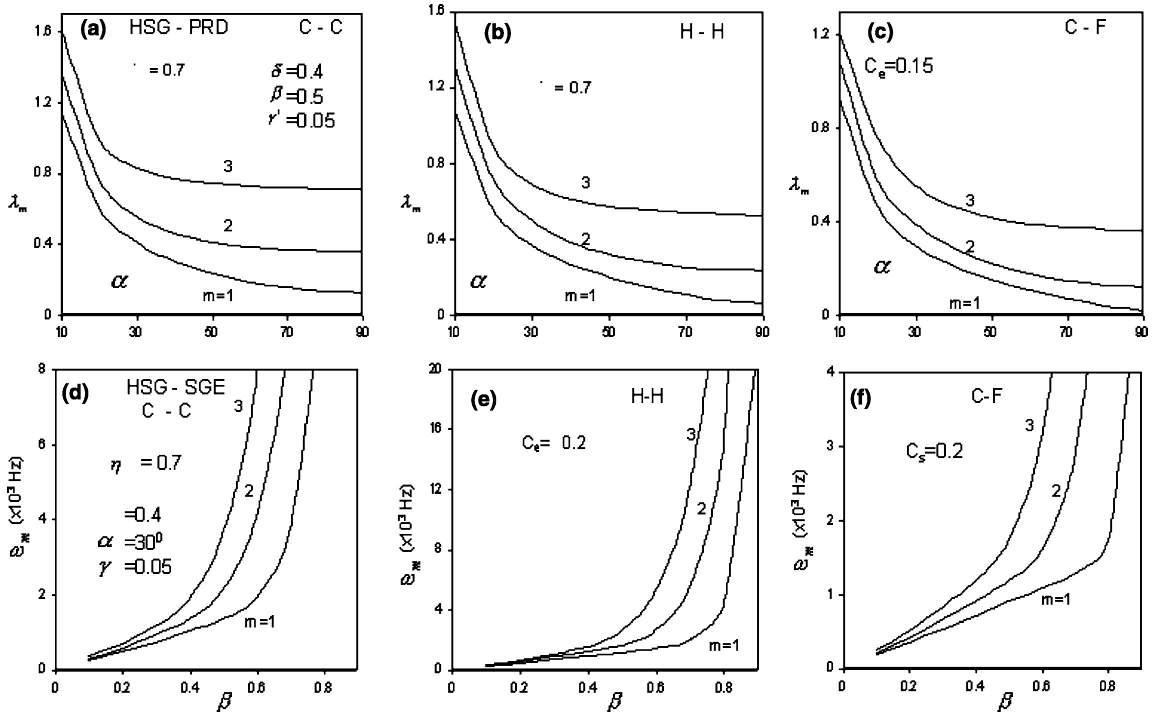

Fig. 6. (a)–(c) Effect of cone angle on frequency parameter: linear and exponential variations in thickness of layers. (d)–(f) Effect of length of cone on frequency parameter: linear, exponential and sinusoidal variations in thickness.

assumed. When the cone angle α varies, $r_a = a \sin\alpha$ also varies, and hence $\gamma = h(a)/r_a$ cannot be held constant. Instead, another parameter $\gamma' = h(a)/a$ is considered. $\gamma = \gamma' \operatorname{cosec}\alpha$.

At the outset the frequency parameter values are found to decrease with increasing cone angle. The decrease is rapid and almost constant up to $\alpha = 20°$ for all cases considered. The same characteristic pattern of changes of frequencies with α is observed when the layers of the shells are varying in thickness exponentially (Fig. 6(c)).

Fig. 6(d)–(f) describe the influence of length ratio of a HSG-SGE cone on its angular frequencies of variation. Since λ is a function of the length ℓ of the shell by definition, it may not be meaningful to study the variation of λ with β. Instead, we study the relation between ω, the angular frequency and β. Since some length parameter must be given a specific value in such cases, we taken $h(a) = 1$ cm for all cases considered.

As a common feature it is seen that the frequencies monotonically increase with increase in β i.e., with decreasing cone-length. Further, the increase is gradual and steady up to some value of β, and rapid afterwards. The rate of gradual change is higher, and the rapid increase in ω starts earlier, for higher modes. With decreasing conicity the range of almost-linear part of gradual increase in frequency increases. As expected, for very short shells ($\beta > 0.8$), frequencies are very high.

In Figs. 7–9 the influence of the nature of variation of thickness of the layers of the shell on its vibrational behaviour is studied. An HSG-PRD shell held under the three types of boundary conditions with the three types of variation in thickness of layers is considered, with $\alpha = 30°$, $\beta = 0.5$ and $\gamma = 0.05$. Fig. 7 relates to linear variation in thickness of layers. When the taper parameter, $\eta = 1$, the thickness is constant. The thickness at the larger end of the cone is larger or smaller than the thickness at the smaller end, according as $\eta \lessgtr 1$. Variation of λ_m $(m = 1, 2, 3)$ with respect to η for $0.5 \leqslant \eta \leqslant 2.1$ is studied. It is seen

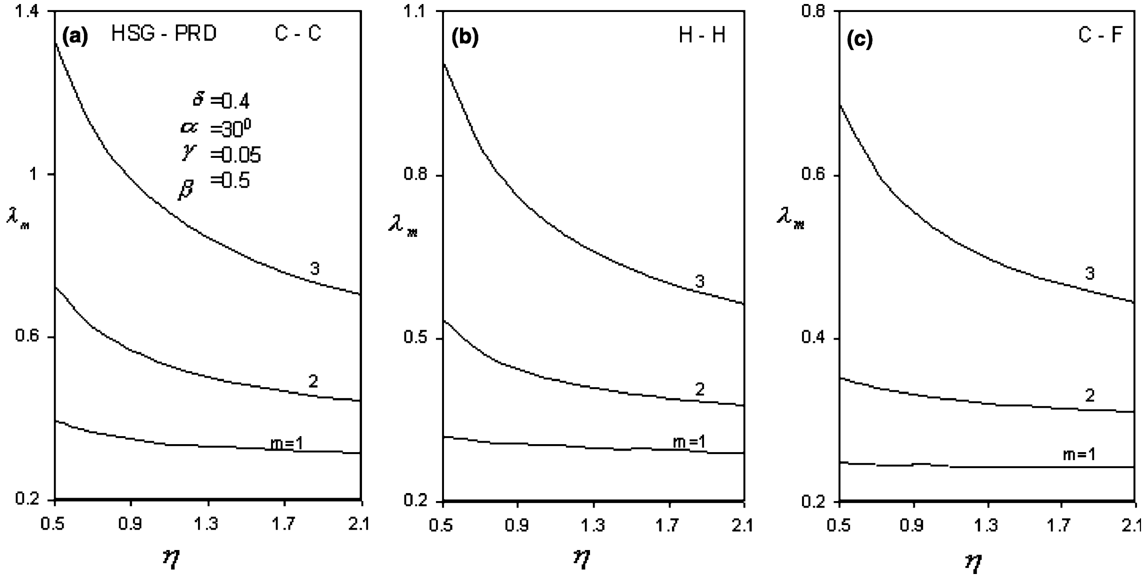

Fig. 7. Effect of taper parameter on frequency parameter.

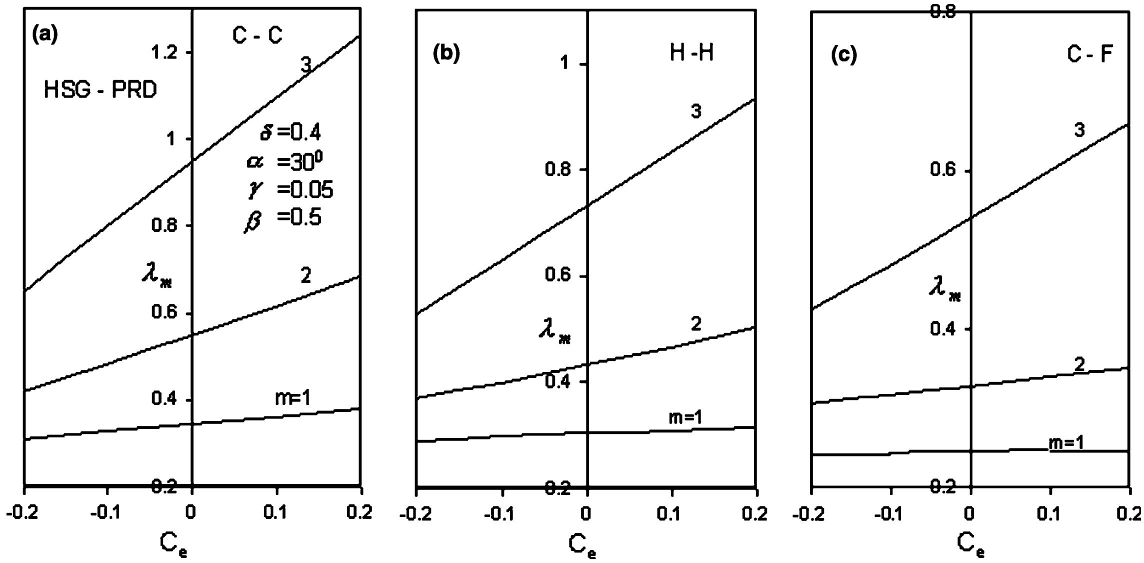

Fig. 8. Effect of coefficient of exponential variation of thickness of layers on frequency parameter.

that λ_m decreases with increase of η. This is in keeping with the fact that smaller the value of η, larger is the thickness, resulting in higher stiffness. The fundamental frequencies are least influenced by taper (In the case of C–F conditions the influence is almost zero) and the influence is higher for higher mode. The percent changes induced in λ_1, λ_2, λ_3 over the range of values of η considered under C–C, H–H and C–F boundary conditions are respectively 25.52%, 65.67%, 90.09%; 9.19%, 41.51%, 80.41% and 1.92%, 13.91%, 55.33%

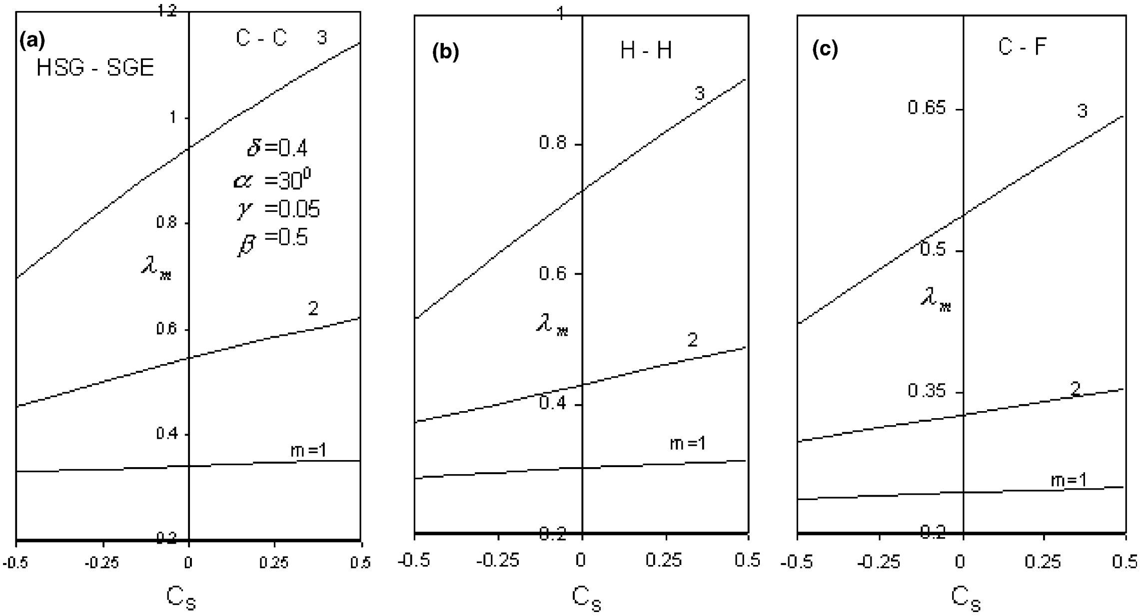

Fig. 9. Effect of coefficient of sinusoidal variation of thickness of layers on frequency parameter.

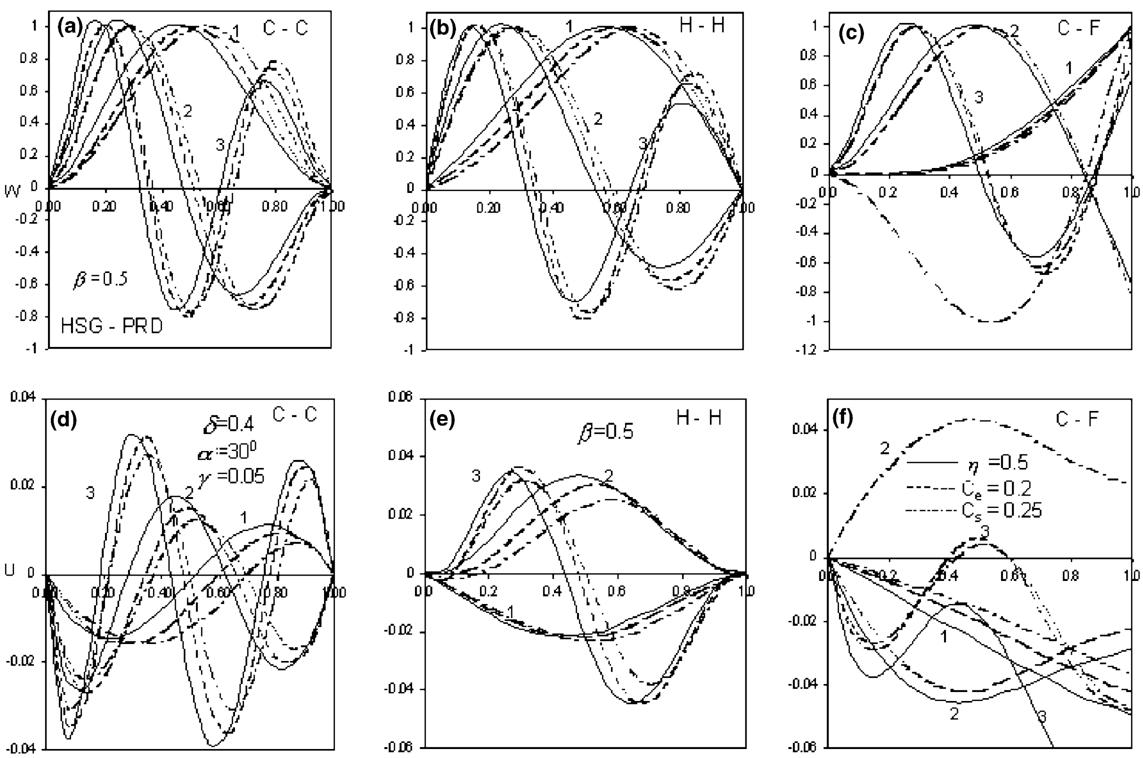

Fig. 10. Some mode shapes of axisymmetric vibration of shells of different types of variation in thickness of layers.

and thus the effect is highest for C–C conditions and lowest for C–F conditions. The curves are convex down for all the cases of Fig. 7.

The effect of exponential variation in thickness of layers is analyzed in Fig. 8. When $C_e = 0$, thickness is uniform. The thickness at the wider end of the cone is higher or lower than the thickness at the other end according as $C_e \lessgtr 0$. This explains why the frequencies are highest and lowest at $C_e = \pm 0.25$, respectively. As in the case of linear variation of thickness, λ_1 increases with mode number. Frequencies increase almost directly as C_e. The percent increases in λ_1, λ_2, λ_3 over the range of values of C_e considered under C–C, H–H and C–F boundary conditions, are, respectively, 22.13%, 63.15%, 92.01%; 8.71%, 36.3%, 77.59% and 2.01%, 14.79%, 56.26%.

The effects of sinusoidal variation in thickness of layers on frequency parameters are studied in Fig. 9. These effects are almost similar to those due to the exponential variation just discussed. Here the coefficient of thickness variation is considered over the range $[-0.5, 0.5]$.

The percent increase in λ_1, λ_2, λ_3 over this range of C_s are, respectively, 6.65%, 37.13%, 64.97% for C–C conditions, 9.34%, 31.63%, 7.92% for H–H conditions and 5.2%, 18.75%, 52.93% for C–F conditions.

Fig. 10 exhibits mode shapes of shells of HSG-PRD layers with $\delta = 0.4$, $\alpha = 30°$, $\beta = 0.5$ and $\gamma = 0.05$, under all the three types of boundary conditions usually considered, and for the first three meridional modes $m = 1, 2, 3$. All the three types of variation of thickness of layers are considered, with $\eta = 0.5$, $C_e = 0.2$ and $C_s = 0.25$, for the respective cases, as indicated. Both the U-and W-displacements are normalized with respect to the maximum W-displacement.

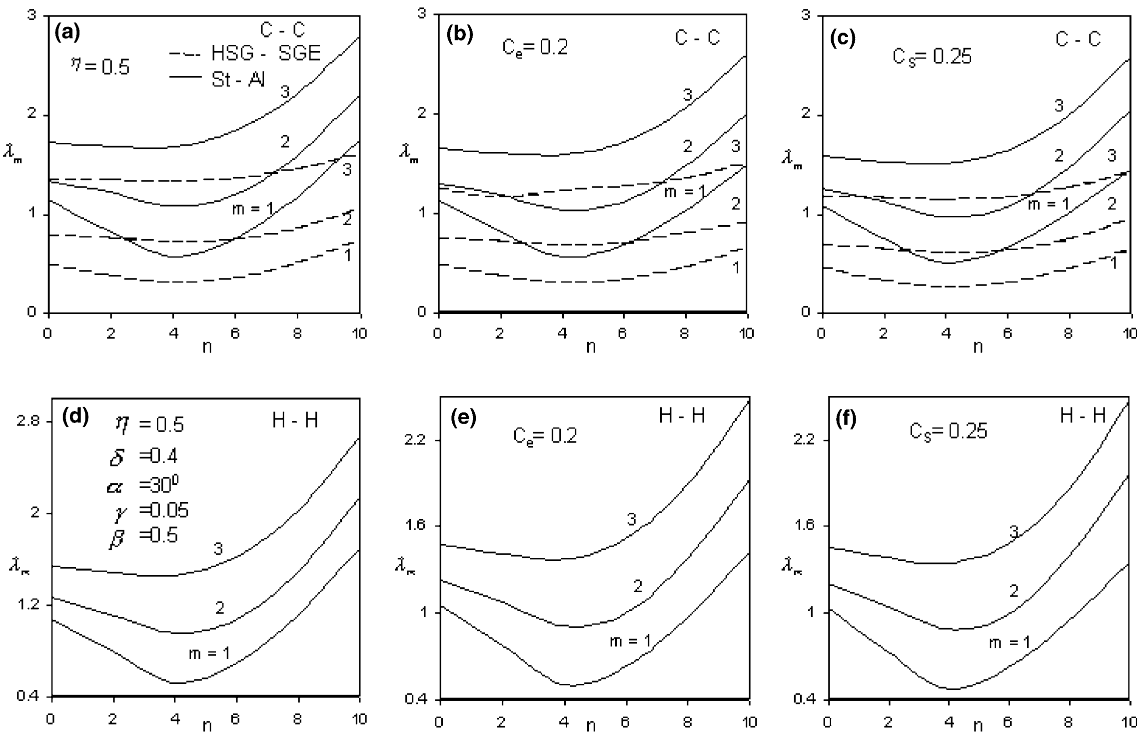

Fig. 11. Effect of circumferential mode number on frequency parameter for different types of variation in thickness of layers.

5.2. Asymmetric vibrations

If $n \neq 0$, then $v \neq 0$ and asymmetric vibrations manifest. Now n, a positive integer, is an additional parameter which is expected to influence the vibrational characteristics of the shell. The values of frequency parameter for the asymmetric vibration of the reduced case of a homogeneous conical shell of constant thickness were obtained for $\alpha = 60°$, $v = 0.3$, $n = 4$, $\beta = 0.25, 0.50, 0.75$ and $\gamma = 0.01/\beta$, and compared, in Table 2 with corresponding results of Irie et al. (1984). This comparison provide a conviction on the correctness of the method and results of the current work.

Fig. 11(a)–(f) shows the manner of variation of the frequency parameter with reference to the circumferential mode number n. We considered $n = 0$ (2) 10. A shell of HSG-SGE lamination and another of St–Al lamination under C–C boundary conditions are considered in Fig. 11(a)–(c). Fig. 11(d)–(f) relates to a shell of St–Al lamination held under H–H boundary conditions. All the shells have semicone angle of 30°, with $\beta = 0.5$, $\gamma = 0.05$ and $\delta = 0.4$. All the three types of variation in thickness of layers are considered, as indicated in the diagrams. It is seen that all the frequency parameter values decreases upto $n = 4$ or 5 and then increase at a faster pace. The curvature at the turning points seem to be greater for lower modes. Fig. 11(a)–(c) shows the effect of n on λ is lower for HSG-SGE shell than for the St–Al shells. The absolute and relatives differences between the maximum and minimum values of λ, caused in the range of values of n considered, is more in the case of C–C boundary conditions than in the case of H–H boundary conditions. The kind of thickness variation in layers does not seem to greatly affect the nature of variation of λ with n.

The effect of the cone angle on frequency parameter is studied in Fig. 12 for two asymmetric vibrations corresponding to $n = 4$ and $n = 8$. The shell considered throughout is made up of HSG-SGE layers, with

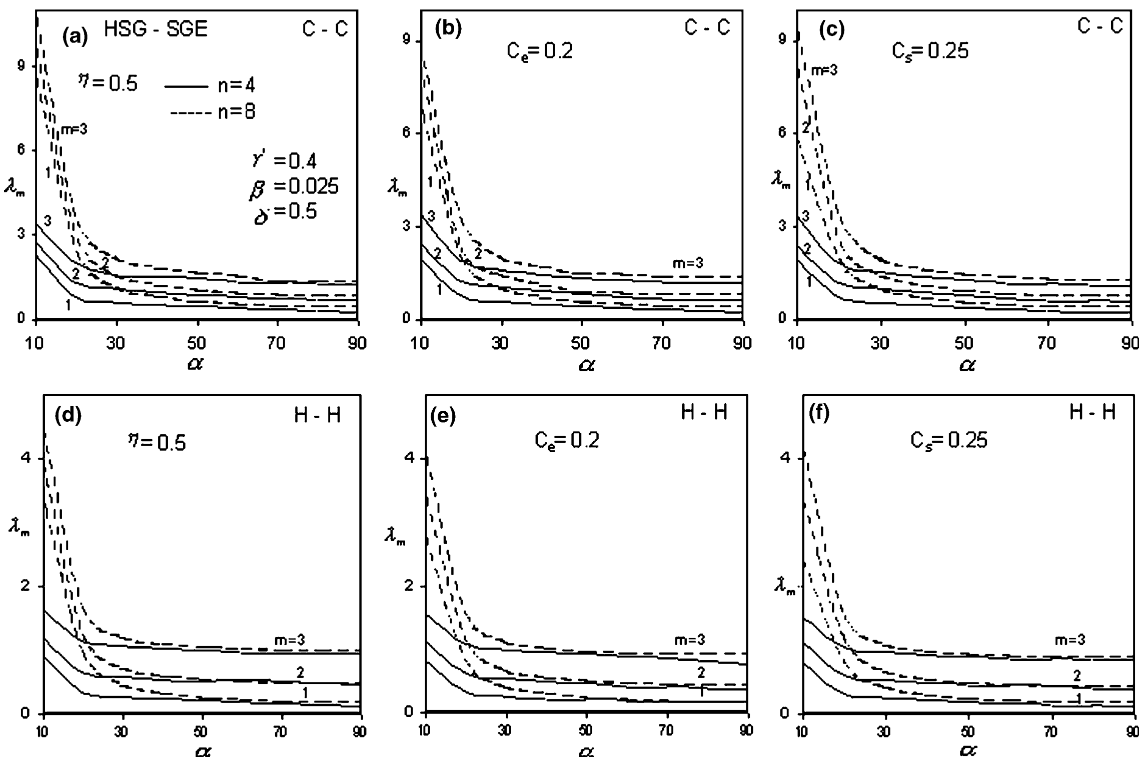

Fig. 12. Effect of cone angle and circumferential mode number on frequency parameter.

medium length ($\beta = 0.5$), $\gamma' = 0.025$ and $\delta = 0.4$. Three thickness variations and two types of boundary conditions are considered. It is commonly found that as α increases from a low value up to about 20°, the values of λ come down almost steadily and rapidly and then continues to decrease, with a sudden decrease in the rapidity, almost becoming a constant as α approaches 90° (the shell becoming a plate). Similar phenomenon was observed in the case of axisymmetric vibrations ($n = 0$) also. While the circumferential mode number affects the slopes of the first parts of these curves (the lines for $n = 8$ are steeper than those for $n = 4$), it does not seem to affect the second parts of these curves significantly. The boundary conditions and types of variation in layer thickness also do not seem to influence the variation of λ with α and n. The variation of frequencies (in Hz) with respect to the length ratio and the circumferential wave number is clarified in Figs. 13 and 14. As for the case of axisymmetric vibrations, for the other values of n also ω increases monotonically, gradually up to some value of β, and then steeply. The change takes place smoothly in the interval $0.4 < \beta < 0.8$. The effects of the two values of n considered do not seem to be significantly different qualitatively, both for the HSG-SGE shell under C–C condition (Fig. 13), and the St–Al shell under H–H condition (Fig. 14).

Fig. 15 exhibits the U, V and W mode shapes, upto three modes, for the circumferential mode number $n = 4$ for the HSG-SGE shell with $\alpha = 30°$, $\beta = 0.5$, $\gamma = 0.5$ and $\delta = 0.4$. All the three types of variations in thickness are considered for the shown thickness parameter values. Transverse displacements are seen to be predominant. The circumferential and extensional displacements follow in that order. The displacements have been normalized with respect to maximum W.

The extensional modes always have one extra node, as also was seen in the case axisymmetric case. Though the nature of variations in thickness of layers induce differences in mode shapes, the general mode

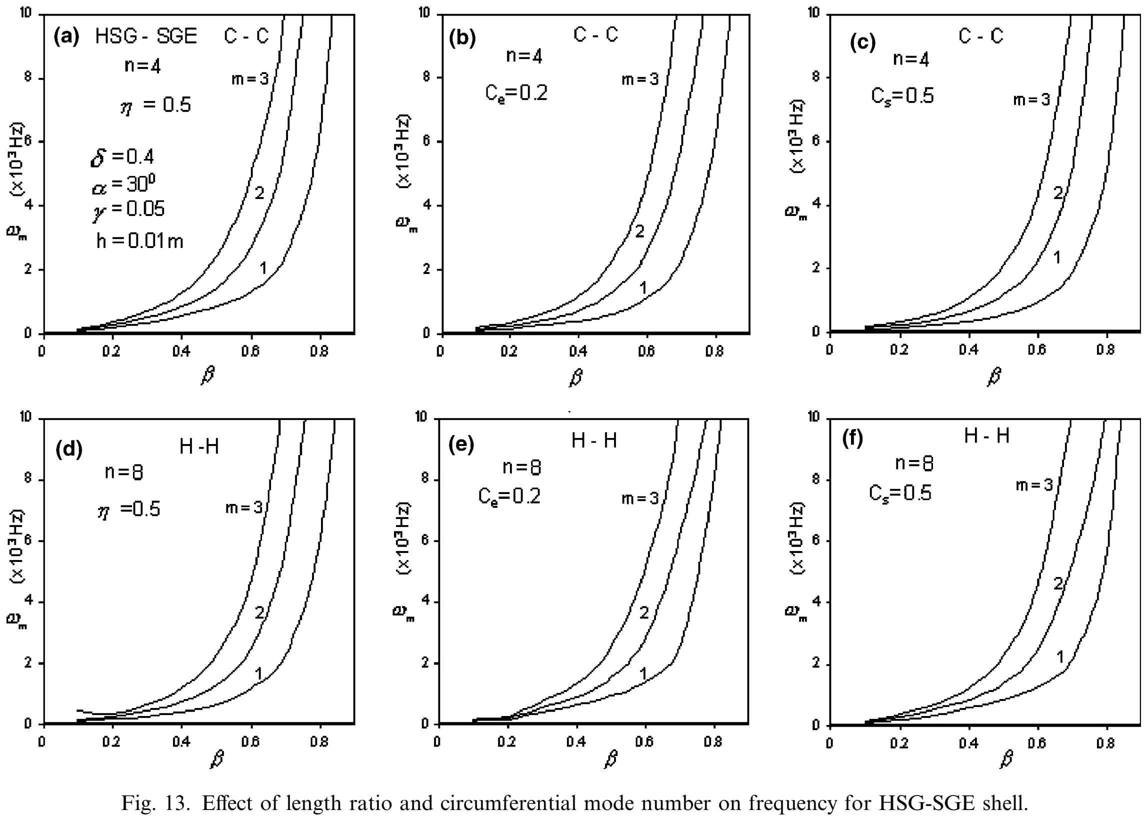

Fig. 13. Effect of length ratio and circumferential mode number on frequency for HSG-SGE shell.

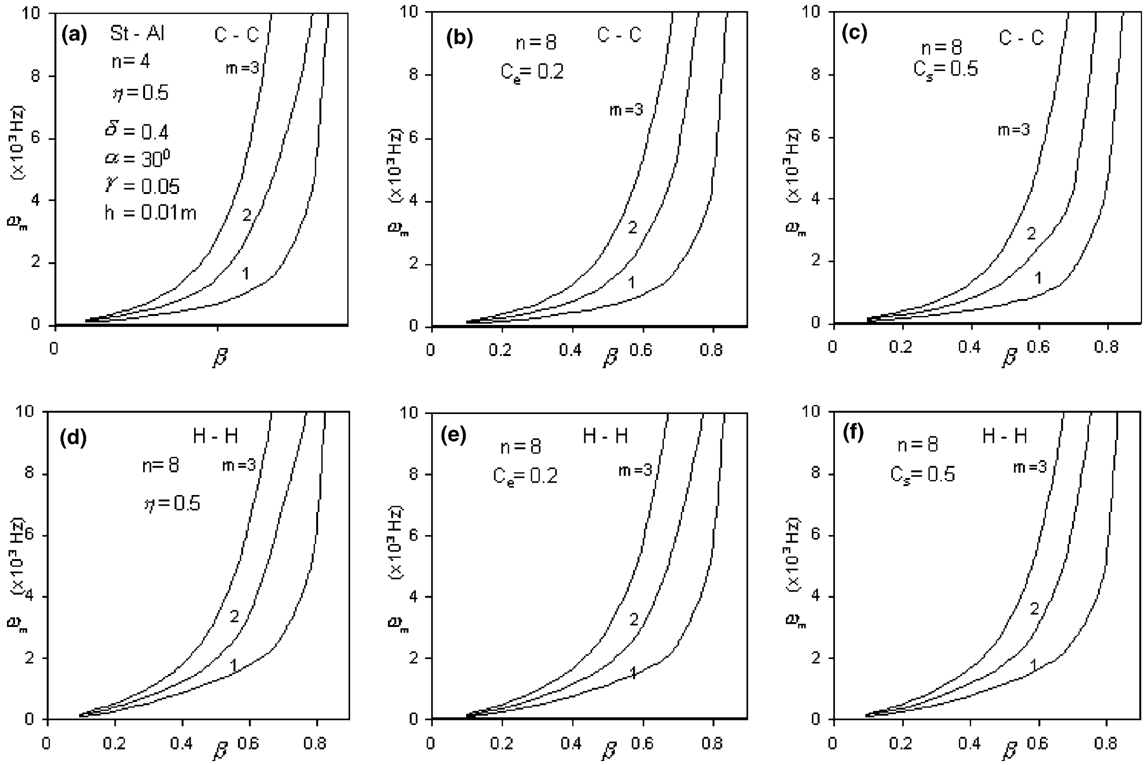

Fig. 14. Effect of length ratio and circumferential mode number on frequency for St–Al shell.

patterns for all the types of thickness variations considered, under the same type of boundary conditions, are seen to be alike.

6. Conclusions

Layering of shell walls with different materials generally results in natural frequencies being considerably different from those of homogeneous shells of any one of the layer materials. These frequencies vary with the relative thickness of layers, the constituent layers, the nature of variation of their thickness, the cone angle, the length ratio of the cone, the ratio of the thickness to radius or length and the boundary conditions. The effect of neglecting the bending-stretching, coupling is generally to raise the frequencies, but quite small for all practical purposes within the realms of thin shell theory. Increasing conicity tends to decrease the frequencies in all cases. This effect is felt more for lower modes and less for shorter shells. This also must be of great significance to designers. The decrease in the length of the cone, while the other characteristics are not altered, results in rise in the frequencies. This effect is higher for higher modes. For any particular set of geometric parameters, the frequencies increase with the overall thickness of the shell wall. All these effects follow the pattern of the general behaviour of homogeneous thin shells.

The nature of variation in thickness of layers considerably affect the natural frequencies of axisymmetric as well as asymmetric vibrations. The effect of taper ratio in the case of linear variation in thickness, for example, is high for very short shells and for very small values of the taper ratio. The effect is felt to a

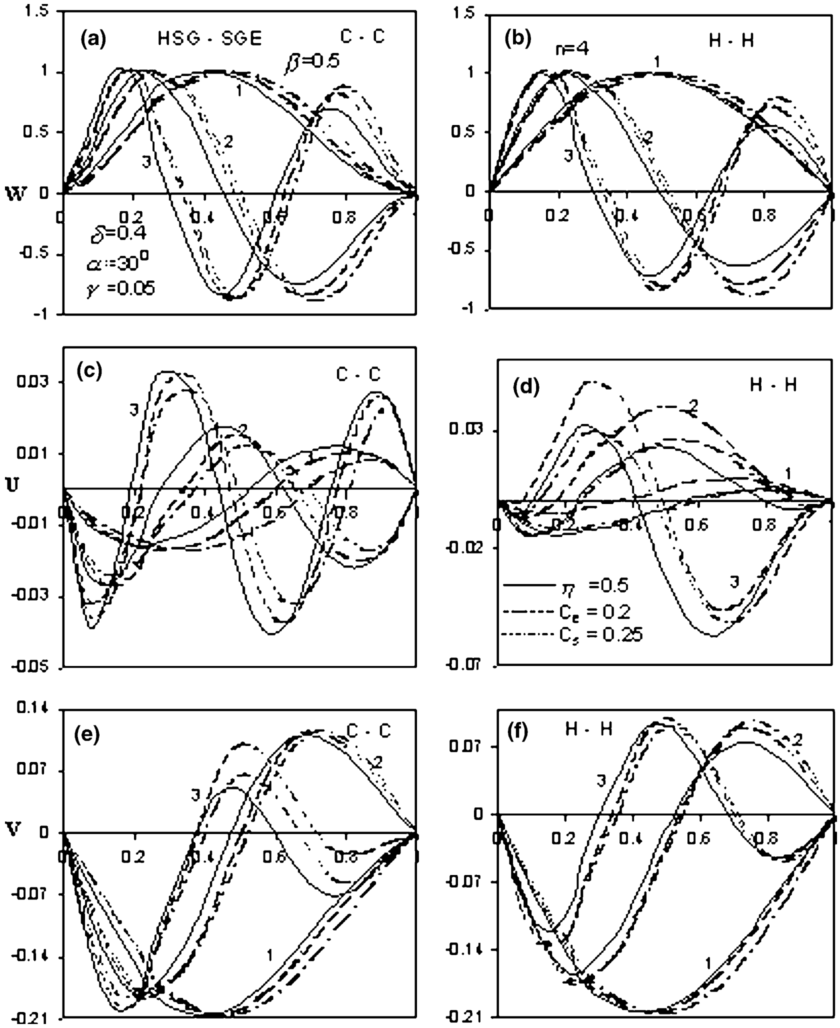

Fig. 15. Some mode shapes of asymmetric vibrations for $n = 4$.

greater extent for a wider cone. In the case of exponential and sinusoidal variations in thickness of layers, the frequencies increase almost proportional to the corresponding coefficients of variation C_e and C_s. The rate of increase is higher for higher meridional modes.

The effect of increasing the cone angle is to decrease the frequencies, whereas the shortening of the length results in their increase. These patterns are similar for all kinds of variations in thickness of layers considered.

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Appendix A. The Equilibrium equations for conical shell frusta of variable thickness

The equilibrium equations are

$$\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}(xN_x) + n\mathrm{cosec}\alpha\frac{N_{x\theta}}{x} - \frac{N_\theta}{x} + g(x)R_0\omega^2 U = 0 \tag{A.1}$$

$$-n\mathrm{cosec}\alpha\frac{N_\theta}{x} + \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}(xN_{x\theta}) + \frac{N_{x\theta}}{x} - \frac{n\mathrm{cosec}\alpha\cot\alpha}{x^2}M_\theta + \frac{\cot\alpha}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}(xM_{x\theta}) + \frac{\cot\alpha}{x^2}M_{x\theta} + g(x)R_0\omega^2 V = 0 \tag{A.2}$$

$$\frac{1}{x}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(xM_x) - n^2\mathrm{cosec}^2\alpha\frac{M_\theta}{x^2} + \frac{2n\mathrm{cosec}\alpha}{x}\frac{\mathrm{d}}{\mathrm{d}x}(M_{x\theta}) - \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}(M_\theta) + \frac{2n\mathrm{cosec}\alpha}{x^2}M_{x\theta} - \frac{\cot\alpha}{x}N_\theta + g(x)R_0\omega^2 W = 0 \tag{A.3}$$

where

$$(N_x, N_\theta, N_{x\theta}) = \int_z (\sigma_x, \sigma_\theta, \sigma_{x\theta})\,\mathrm{d}z \tag{A.4}$$

$$(M_x, M_\theta, M_{x\theta}) = \int_z (\sigma_x, \sigma_\theta, \sigma_{x\theta})z\,\mathrm{d}z \tag{A.5}$$

and

$$R_0 = \sum_k \rho^{(k)}[z_k(a) - z_{k-1}(a)] = \sum_k \rho_k h_k(a) \quad \text{is the inertial coefficient.} \tag{A.6}$$

Appendix B

The differential operators L_{ij} appearing in Eq. (8) are

$$L_{11} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{g'}{g} + \frac{1}{x}\right)\frac{\mathrm{d}}{\mathrm{d}x} + s_2\frac{g'}{xg} - \frac{1}{x^2}(s_3 + s_{10}n^2\mathrm{cosec}^2\alpha) + \lambda'^2 \tag{B.1}$$

$$L_{12} = n\mathrm{cosec}\alpha\left[\frac{1}{x}(s_2 + s_{10}) + (s_5 + 2s_{11})\cot\alpha\frac{g}{x^2}\right]\frac{\mathrm{d}}{\mathrm{d}x} + n\mathrm{cosec}\alpha\left[s_2\frac{g'}{xg} - \frac{s_{10}}{x^2} - \frac{s_3}{x^2} + s_5\cot\alpha\left(\frac{2g''}{x^2} - \frac{g}{x^3}\right) + 2s_{11}\frac{g}{x^3}\cot\alpha - s_6\cot\alpha\frac{g}{x^3}\right] \tag{B.2}$$

$$L_{13} = -s_4 g\frac{\mathrm{d}^3}{\mathrm{d}x^3} - s_4\left(\frac{g}{x} + 2g'\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left[s_6\frac{g}{x^2} + \frac{g}{x^2}(s_5 + 2s_{11})n^2\mathrm{cosec}^2\alpha - 2s_5\frac{g'}{x} + s_2\frac{\cot\alpha}{x}\right]\frac{\mathrm{d}}{\mathrm{d}x} + \left(s_2\frac{g'}{xg} - \frac{s_3}{x^2}\right)\cot\alpha + \left(\frac{2g'}{x^2} - \frac{g}{x^3}\right)s_5 n^2\mathrm{cosec}^2\alpha - s_6 n^2\mathrm{cosec}^2\alpha\frac{g}{x^3} - 2s_{11}n^2\mathrm{cosec}^2\alpha\frac{g}{x^3} \tag{B.3}$$

$$
\begin{aligned}
L_{21} = &-n\operatorname{cosec}\alpha\left[\frac{s_2}{x} + \frac{s_{10}}{x} + s_5 \cot\alpha \frac{g}{x^2} + s_{11} \cot\alpha \frac{g}{x^2} -\right]\frac{\mathrm{d}}{\mathrm{d}x} \\
&- n\operatorname{cosec}\alpha\left[\frac{s_3}{x^2} + \frac{s_{10}}{x^2} + s_6 \cot\alpha \frac{g}{x^3} + s_{11}\cot\alpha\left(-\frac{g}{x^2} - \frac{2g'}{x^2}\right) + s_{10}\frac{g'}{xg}\right]
\end{aligned} \tag{B.4}
$$

$$
\begin{aligned}
L_{22} = &\left[\left(3s_{11} + 2s_{12}\cot\alpha\frac{g}{x}\right)\cot\alpha\frac{g}{x} + s_{10}\right]\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left[s_{10}\left(\frac{1}{x} + \frac{g'}{g}\right) + s_{11}\left(\frac{g}{x^2} + \frac{6g'}{x}\right)\cot\alpha + 6s_{12}\cot^2\alpha\frac{gg'}{x^2}\right]\frac{\mathrm{d}}{\mathrm{d}x} \\
&- \left[s_{11}\left(\frac{g}{x^3} + \frac{6g'}{x^2}\right) + \left(2s_6 + s_9\cot\alpha\frac{g}{x}\right)n^2\operatorname{cosec}^2\alpha\frac{g}{x^3}\right]\cot\alpha - s_3\frac{n^2\operatorname{cosec}^2\alpha}{x} - s_{10}\left(\frac{g'}{xg} + \frac{1}{x^2}\right) \\
&- 6s_{12}\cot^2\alpha\frac{gg'}{x^3} + \lambda'^2
\end{aligned} \tag{B.5}
$$

$$
\begin{aligned}
L_{23} = &\, n\operatorname{cosec}\alpha\frac{g}{x}\left[s_5 + (s_8 + 2s_{12})\cot\alpha\frac{g}{x} + 2s_{11}\right]\frac{\mathrm{d}^2}{\mathrm{d}x^2} \\
&+ \frac{n\operatorname{cosec}\alpha}{x}\left[\frac{s_6 g}{x} + 4s_{11}g' + \cot\alpha\left(\frac{s_9 g^2}{x^2} + 6s_{12}\frac{gg'}{x}\right)\right]\frac{\mathrm{d}}{\mathrm{d}x} \\
&- n\operatorname{cosec}\alpha\left[6s_{12}\cot\alpha\frac{g'g}{x^3} + 4s_{11}\frac{g'}{x^2} + \left(s_9 n^2\operatorname{cosec}^2\alpha\frac{g^2}{x^4} + \frac{s_3}{x^2}\right)\cot\alpha + s_6\frac{g}{x^3}(\cot^2\alpha + n^2\operatorname{cosec}^2\alpha)\right]
\end{aligned} \tag{B.6}
$$

$$
\begin{aligned}
L_{31} = &\, s_4 g\frac{\mathrm{d}^3}{\mathrm{d}x^3} + \frac{s_4}{x}(4xg' + 2g)\frac{\mathrm{d}^2}{\mathrm{d}x^2} \\
&+ \left[\frac{s_4}{x}\left(4g' + \frac{2xg'^2}{g} + 2xg''\right) - s_2\frac{\cot\alpha}{x} + 2s_5\frac{g'}{x} - s_6\frac{g}{x^2} - n^2\operatorname{cosec}^2\alpha\frac{g}{x^2}(s_5 + 2s_{11})\right]\frac{\mathrm{d}}{\mathrm{d}x} \\
&+ \frac{s_5}{x}\left(\frac{2g'^2}{g} + 2g''\right) - \frac{s_6}{x^2}\left(2g' - \frac{g}{x}\right) - s_3\frac{\cot\alpha}{x^2} - \frac{n^2\operatorname{cosec}^2\alpha}{x^2}\left(\frac{g}{x}s_6 + 4s_{11}g'\right)
\end{aligned} \tag{B.7}
$$

$$
\begin{aligned}
L_{32} = &\, n\operatorname{cosec}\alpha\frac{g}{x}\left[s_5 + 2s_{11} + 4s_{12} + \cot\alpha\frac{g}{x}(s_8 + 4s_{12})\right]\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{n\operatorname{cosec}\alpha}{x}\left[4s_5 g' + s_8\cot\alpha\left(\frac{6gg'}{x} - \frac{2g^2}{x^2}\right)\right. \\
&\left. + \frac{2g}{x^2}\left(s_{11} + 2s_{12}\cot\alpha\frac{g}{x}\right) + \frac{2}{x}\left(2s_{11}g' - s_{11}\frac{g}{x} - 4s_{12}\cot\alpha\frac{gg'}{x}\right) - \frac{g}{x^2}\left(s_9\cot\alpha\frac{g}{x} + s_6\right)\right]\frac{\mathrm{d}}{\mathrm{d}x} \\
&- s_6 n\operatorname{cosec}\alpha\frac{g}{x^3}(\cot^2\alpha + n^2\operatorname{cosec}^2\alpha) + \frac{n\operatorname{cosec}\alpha\cot\alpha}{x^2}\left[s_8\left(3gg' + \frac{2g^2}{x^2} + 6g'^2 - \frac{6gg'}{x}\right)\right. \\
&\left. - s_3 - s_9 n^2\operatorname{cosec}^2\alpha\frac{g^2}{x^2}\right] + \frac{n\operatorname{cosec}\alpha}{x}\left[s_5\left(2g'' + \frac{2g'^2}{g}\right) - 4s_{11}\frac{g'}{x} + 4s_{12}\cot\alpha\frac{g^2}{x^3} - 12s_{12}\cot\alpha\frac{gg'}{x^2}\right. \\
&\left. + s_6\left(\frac{g}{x^2} - \frac{2g'}{x}\right) + s_9\cot\alpha\frac{g}{x^2}\left(\frac{2g}{x} - 3g'\right)\right]
\end{aligned} \tag{B.8}
$$

$$
\begin{aligned}
L_{33} = {} & -s_7 g^2 \frac{d^4}{dx^4} - \frac{s_7}{x}(6xgg' + 2g^2)\frac{d^3}{dx^3} + \left[\frac{3s_8 gg'}{x} + \frac{s_7 gg^2}{x^2} - \frac{s_7}{x}(6gg' + 3xgg'' + 6xg'^2)\right. \\
& \left. + s_8\left(n^2 \mathrm{cosec}^2\alpha \frac{g^2}{x^2} - \frac{6gg'}{x}\right) + 2s_5 \cot\alpha \frac{g}{x} + (s_8 + 4s_{12})\frac{g^2}{x^2} n^2 \mathrm{cosec}^2\alpha\right]\frac{d^2}{dx^2} \\
& + \left[4s_5 \cot\alpha \frac{g'}{x} + s_9\left(\frac{3gg'}{x^2} - \frac{g^2}{x^3}\right) - \frac{s_8}{x}\left(3gg' + 6g'^2 + \frac{2n^2 \mathrm{cosec}^2\alpha g^2}{x^2} - 6n^2 \mathrm{cosec}^2\alpha \frac{gg'}{x}\right)\right. \\
& \left. - \frac{s_{12}}{x}\left(\frac{4g^2}{x^2} - \frac{6gg'}{x}\right) n^2 \mathrm{cosec}^2\alpha\right]\frac{d}{dx} + 2s_{12} n \mathrm{cosec}\alpha\left(\frac{4g^2}{x^4} - \frac{6gg'}{x^3}\right) + s_6 \cot\alpha\left(\frac{g}{x^3} - \frac{2g'}{x^2}\right) \\
& + s_9 n^2 \mathrm{cosec}^2\alpha\left(\frac{2g^2}{x^4} - \frac{3gg'}{x^3} - n^2 \mathrm{cosec}^2\alpha \frac{g^2}{x^4}\right) - s_3 \frac{\cot^2\alpha}{x^2} - 2s_6 n^2 \mathrm{cosec}^2\alpha \cot\alpha \frac{g}{x^3} \\
& + s_5 \cot\alpha\left(\frac{2g'^2}{xg} + \frac{2g''}{x}\right) - 4s_{12} n^2 \mathrm{cosec}^2\alpha \frac{g^2}{x^4} + \frac{s_8 n^2 \mathrm{cosec}^2\alpha}{x^2}\left(3gg' + 6g'^2 + \frac{2g^2}{x^2} - \frac{6gg'}{x}\right) + \lambda'^2
\end{aligned}
\tag{B.9}
$$

in which

$$
s_2 = \frac{A_{12}^c}{A_{11}^c}, \quad s_3 = \frac{A_{22}^c}{A_{11}^c}, \quad s_4 = \frac{B_{11}^c}{A_{11}^c}, \quad s_5 = \frac{B_{12}^c}{A_{11}^c}, \quad s_6 = \frac{B_{22}^c}{A_{11}^c}, \quad s_7 = \frac{D_{11}^c}{A_{11}^c}, \quad s_8 = \frac{D_{12}^c}{A_{11}^c},
$$
$$
s_9 = \frac{D_{22}^c}{A_{11}^c}, \quad s_{10} = \frac{A_{66}^c}{A_{11}^c}, \quad s_{11} = \frac{B_{66}^c}{A_{11}^c} \quad s_{12} = \frac{D_{66}^c}{A_{11}^c} \tag{B.10}
$$

and

$$
\lambda'^2 = \frac{R_0 \omega^2}{A_{11}^c} \quad \text{is a frequency parameter.} \tag{B.11}
$$

where R_0 is the inertial coefficient defined in Appendix A and A_{11}^c is the elastic coefficient representing the extensional rigidity.

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